

On compressive radial tidal forces

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Abstract

Radial tidal forces can be compressive instead of disruptive, a possibility that is frequently overlooked in high level physics courses. For example, radial tidal compression can emerge in extended stellar systems containing a smaller stellar cluster. For particular conditions the tidal field produced by this extended mass distribution can exert on the cluster it contains compressive effects instead of the common disruptive forces. This interesting aspect of gravity can be derived from standard relations given in many textbooks and introductory courses in astronomy and can serve as an opportunity to look closer at some aspects of gravitational physics, stellar dynamics, and differential geometry. The existence of compressive tides at the center of huge stellar systems might suggest new evolutionary scenarios for the formation of stars and primordial galactic formation processes.

I. INTRODUCTION

Tides are the manifestation of a gradient of the gravitational force field induced by a mass above an extended body or a system of particles. In the solar system tidal perturbations act on compact bodies such as planets, moons, and comets.¹ On larger scales than the solar system, as in a galactic or cosmological context, we frequently deal with tidal deformations (and eventually even disruptions) of a stellar cluster by a galaxy or in galactic encounters. The study of tidal fields in the most simple cases can be done by analytically approximating the potential field of a nonspherical mass distribution by a spherical (that is, a point mass) potential. This approximation sometimes leads to gross estimates and is inappropriate for the analysis of tidal fields within mass distributions.

Stars can be considered as particles with a long mean free path, and the clusters they form as collisionless fluid dynamical systems. On galactic or cosmological scales a huge mass distribution can interact gravitationally with a smaller system that travels through it or is inside of it, without any direct contact taking place between its constituents. For example, think of a globular cluster passing through a disk or near the center of a galaxy, or a dust cloud orbiting inside a star cluster. The tidal field produced by the larger system, acting on the smaller system it contains, needs to be computed inside the mass distribution. Tidal forces can in these cases behave very differently than in the cases encountered in the solar system.

Although numerical simulations are usually necessary to study an extended mass distribution, the analytic study of some representative, not too approximate models is sometimes possible. The effect of tides due to an extended mass distribution has been an active field of research in the last two decades, especially since high speed numerical computers became available. Typical cases of interest are galactic encounters,² globular clusters under the influence of the galactic mass distribution,^{3,4} and Oort cloud perturbations by the galactic field.⁵ However, most of these investigations were on disruptive tidal forces.

We are accustomed to think of gravitation as a force field that decreases rapidly with the distance from the field source. This dependence is not necessarily true inside a mass distribution producing the field where the force can increase with the distance from the center of mass. A trivial example is a spherical homogeneous mass distribution with radius R and constant density ρ_0 . To an outside observer at distance $r > R$ the force field is

$F(r) = -GM/r^2$, with $M = \rho_0 \frac{4}{3}\pi R^3$, and the tidal force $\partial F(r)/\partial r = 2GM/r^3 > 0$, which because it is positive is responsible for the typical tidal bulges, that is, disruptive effects. If we move inside the same distribution, we obtain $F(r) = -GM(r)/r^2 = -\frac{4}{3}\pi G\rho_0 r$, but $\partial F/\partial r = -\frac{4}{3}\pi G\rho_0 < 0$. The tidal field has become compressive because inside the body the gravitational strength increases with the distance from the center of the sphere.

A constant density fluid system in gravitational equilibrium is an unrealistic case. The effect of the compressive nature of tidal fields has been studied in the context of disk shocking of star clusters.⁶ Fast encounters with the galactic disk or bulge cause gravitational shocks where the entire cluster expands, but with a short period of contraction immediately after the shock. Another attempt to understand this effect was made by Valluri⁷ who studied compressive tidal heating effects. The effects of the mean tidal field of a cluster of galaxies on the internal dynamics of a disk galaxy traveling through it was discussed, and the disk experiences a compressive tidal field within the core of the cluster. Das and Jog⁸ investigated the same effect for compressed molecular cloud dynamics. They suggested that the compressive tidal field in the center of flat core early type galaxies and ultraluminous galaxies compresses molecular clouds producing the dense gas observed in the center of these galaxies.

The nature of the radial compressive tide remains largely unexplored. It is important not only for its physical effects, but because it is a fundamental aspect of gravitation itself. In the following we formulate a tidal field theory in which this effect arises naturally and investigate its relevance in some real systems such as stellar globular clusters and galactic centers.

II. THE CONSTRUCTION AND MEANING OF THE NEWTONIAN TIDAL TENSOR

The Newtonian tidal tensor is a useful tool for the study of tidal fields. The use of the tidal tensor for this purpose is not new, although not common. It has been employed for example in the study of the angular momentum growth in protogalaxies and more recently for investigating the origin of angular momentum in galaxies through tidal torquing (see for example, Refs. 9 and 10). Here we analyze the tidal field inside a mass distribution and show how the latter naturally contains regions of radial tidal forces sign reversal.

Because the Newtonian tidal tensor is frequently employed in a somewhat obscure way in the literature, it is useful to recall some of its properties and see how it naturally leads to the common tidal textbook formalism.

Once a coordinate system and a frame of reference is defined (typically with its origin at the mass distribution's center of mass) and given the mass distribution's gravitational force field $\mathbf{F}(\mathbf{R})$ (with the vector \mathbf{R} vector at a particle of the tidally perturbed system), the tidal forces arise as the manifestation of the gradient $\nabla\mathbf{F}(\mathbf{R})$. The tidal force induced by a mass distribution on another body or another system of particles is defined as the difference between the gravitational force that the mass exerts at that point, and the mean gravitational force $\langle\mathbf{F}\rangle$ it exerts on the whole body or system

$$\mathbf{F}_t(\mathbf{R}) = \mathbf{F}(\mathbf{R}) - \langle\mathbf{F}\rangle. \quad (1)$$

It can be shown that the mean gravitational force acting on the tidally perturbed system is equivalent to the force acting at its center of mass, \mathbf{R}_{cm} . Hence without any loss of generality, we can write the tidal fields as

$$\mathbf{F}_t(\mathbf{R}) = \Delta\mathbf{F} = \mathbf{F}(\mathbf{R}) - \mathbf{F}(\mathbf{R}_{\text{cm}}). \quad (2)$$

Once the gravitational force field is expressed in terms of the potential $\Phi(\mathbf{R})$ as $\mathbf{F}(\mathbf{R}) = -\nabla\Phi(\mathbf{R})$, \mathbf{F}_t is completely determined if we introduce the second rank symmetric Newtonian tidal force tensor¹¹

$$\tau_{ij} = -\frac{\partial^2\Phi}{\partial x_i\partial x_j}, \quad (3)$$

with $i, j = (1, 2, 3)$ and $x_1 = x$, $x_2 = y$, $x_3 = z$; τ_{ij} is symmetric due to the conservative character of gravitational fields. If we use Einstein's summation convention, the components of the tidal force F_t in its differential form is

$$dF_{t_{x_i}} = \frac{\partial F_i}{\partial x_j} dx^j = -\frac{\partial^2\Phi}{\partial x_i\partial x_j} dx^j = \tau_{ij} dx^j. \quad (4)$$

The tensor τ is the Jacobian of the gravitational force field \mathbf{F} and the negative Hessian matrix of the scalar potential function Φ . These analytic properties of τ give information about the maximum, minimum, and saddle points of the potential surface $\Phi(x, y, z)$.¹² The negative of the trace of τ gives Poisson's equation, which relates the potential Φ to the mass density ρ

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho(x, y, z). \quad (5)$$

Poisson's equation is nothing more than a trace invariant quantity of the tidal tensor matrix.

It is possible to formulate the gravitational potential Φ through Poisson's equation, and therefore the force field of heterogeneous spherical, spheroidal, and disk-like mass distributions.¹³ This formulation is not straightforward, especially for irregular and complex mass distributions. From linear algebra we know that the trace of a Hessian τ of a local surface patch is an invariant quantity under coordinate transformations and that it equals the sum of its eigenvalues, λ_i , that is, $\text{Tr}(\tau) = \sum_i \lambda_i$. Differential geometry defines twice the value of the mean local curvature (the mean value of curvature along the axes) as $H = \frac{1}{2}\text{Tr}(\tau)$, and the Gaussian curvature is defined by the product of the eigenvalues: $K = \prod_i \lambda_i$, or by the determinant $\det(\tau)$. Hence, there seems to be a relation between the Newtonian tidal tensor, the curvature of the potential surface it represents, and mass density and tidal forces, which is reminiscent of the connection between spacetime curvature and matter in general relativity. This relation is not a coincidence.

The Newtonian tidal tensor τ_{ij} is the classical counterpart of the more general fourth rank Riemann spacetime curvature tensor of general relativity, which is why it is called Newtonian. Those familiar with general relativity know that the Riemann tensor, R^i_{kjl} , is also known as the *tidal force tensor*. It can be shown¹⁴ that in the weak field limit the Riemann tensor is the classical Newtonian tidal force tensor, that is, $R^i_{00j} = \tau_{ij}$. Curvature in general relativity is not associated with Newton's inverse square force law, but rather with the relative accelerations of neighboring particles (their geodesic deviation), that is, with tidal forces. We might say that spacetime curvature and tidal forces represent the same physical entity, or to put it in Bondi's words,¹⁵ "Einstein's theory of gravitation has its physical and logical roots . . . in the existence of Newtonian tidal forces."

From this point on the Newtonian tidal tensor will be referred to as the tidal tensor, implying, if not stated otherwise, the classical one.

To become better acquainted with the procedure of the next sections it is useful to recover the main concepts associated with the most typical tidal perturbation – that induced by the spherical potential of a point mass¹⁶ $\Phi^{\text{pm}}(r) = -GM/r$, where M is the mass of the point mass and G the gravitational constant. By applying Eq. (3) the tidal tensor in Cartesian coordinates for an arbitrary frame of reference does not look very elegant, because once we fix the coordinate origin, the tidal tensor is generally non-diagonal for any test particle in a gravitational force field. Because it is a real symmetric matrix, there

must be a frame of reference for which τ^{pm} is diagonal.¹⁷ If we choose a frame of reference where the x -axis passes through the particle's position, corresponding to the radial spherical coordinate, that is, $(x = r, y = 0, z = 0)$, then τ is diagonal and becomes

$$\tau^{\text{pm}} = G \begin{pmatrix} \frac{2M}{x^3} & 0 & 0 \\ 0 & -\frac{M}{x^3} & 0 \\ 0 & 0 & -\frac{M}{x^3} \end{pmatrix} \equiv G \begin{pmatrix} \frac{2M}{r^3} & 0 & 0 \\ 0 & -\frac{M}{r^3} & 0 \\ 0 & 0 & -\frac{M}{r^3} \end{pmatrix}. \quad (6)$$

From Eq. (4) we obtain

$$dF_{t_x}^{\text{pm}} = \tau_{xx}^{\text{pm}} dx = \frac{2GM}{r^3} dx \quad (7a)$$

$$dF_{t_y}^{\text{pm}} = \tau_{yy}^{\text{pm}} dy = -\frac{GM}{r^3} dy \quad (7b)$$

$$dF_{t_z}^{\text{pm}} = \tau_{zz}^{\text{pm}} dz = -\frac{GM}{r^3} dz, \quad (7c)$$

which represents the cubic dependence of the radial force, and the orthogonal longitudinal and latitudinal differential tidal forces, respectively.

The radial tidal force represents a change in the absolute value of the force vector, and the longitudinal and latitudinal vectors represent an angular variation of the components of the central force:

$$\tau_{xx}^{\text{pm}} = \frac{2GM}{r^3} = \frac{\partial \mathbf{F}(r)}{\partial r} \quad (8a)$$

$$\tau_{yy}^{\text{pm}} = \tau_{zz}^{\text{pm}} = -\frac{GM}{r^3} = \frac{\mathbf{F}(r)}{r}. \quad (8b)$$

The sign has an important meaning. The positive sign (in this case that of the radial tide) represents a tensile force and the negative one represents a compressive effect.¹⁸

From Eq. (7a) the radial tidal force induced by a point mass is to first order in x

$$F_{tx}^{\text{pm}} = \frac{2GM}{r_{\text{cm}}^3} \Delta x, \quad (9)$$

where Δx is the distance of the particle undergoing that tidal force from the center of mass of the perturbed body, and r_{cm} is the distance of this center of mass from the tide raising point mass of mass M . Equation (9) is a common textbook expression for tidal forces. From Eq. (9) we recover the celebrated Roche's limit in the usual way

$$\Delta x_{\text{Roche}} = \left(\frac{M_{\text{cm}}}{2M} \right)^{1/3} r_{\text{cm}}. \quad (10)$$

The quantity Δx_{Roche} , also called the tidal radius, represents the distance at which the disruptive radial tidal force on a system of free particles (a perfectly fluid body) of mass M_{cm} exactly opposes the system's self-gravity, $F(r) = -GM_{\text{cm}}/\Delta x^2$, the force that holds it together. Inside Roche's limit this body will be torn apart by tidal forces.

Historically the description of tidal fields beginning with tensor-like objects was less successful than its potential or force field counterpart because a tensor field needs more entities to be visualized (for example, at each point imagine the vectors representing the columns of a matrix), while the concept of a scalar or a vector field is intuitively easier to grasp. But the price paid might be worse because textbooks sometimes present lengthy and complicated calculations to recover Eq. (7).

A. The tidal field of the spherical heterogeneous mass distribution

As a good example of tidal fields inside matter distributions, we study the potential of a heterogeneous spherical mass distribution. We can show¹³ that the resulting spherical potential is given by

$$\Phi^{\text{sph}}(r) = -4\pi G \left[\frac{1}{r} \int_0^r \rho(r') (r')^2 dr' + \int_r^{R_{\text{max}}} \rho(r') r' dr' \right], \quad (11)$$

where $\rho(r)$ is the radial mass density and R_{max} is the maximum extension of the spherical mass distribution (if we assume that no cutoff radius exists we can set $R_{\text{max}} = \infty$). Simple geometric considerations tell us that the radial dependence of the enclosed mass in a sphere of radius r and mass density distribution $\rho(r)$ is

$$M(r) = 4\pi \int_0^r \rho(r') (r')^2 dr', \quad (12)$$

and therefore we write Eq. (11) as

$$\Phi^{\text{sph}}(r) = -\frac{G M(r)}{r} - 4\pi G \int_r^{R_{\text{max}}} \rho(r') r' dr'. \quad (13)$$

The first integral term on the right-hand side of Eq. (11) is the gravitational potential, whose derivative is in accordance with Newton's second theorem, which states that the gravitational force on a body placed outside a spherical shell of mass M is equivalent to the force produced by a point mass at the center of that shell.²⁰ The second integral accounts for the effect of the external mass distribution. We do not necessarily need to know the

radial mass dependence $M(r)$ to calculate τ . The potential Φ or the mass density function $\rho(r)$ alone can give us all the required information.

The application of Eq. (3) to Eq. (11) is not so straightforward because the former has been defined in Cartesian coordinates and the latter in spherical coordinates. We expect that the calculations become easier if we rewrite the tidal tensor in spherical coordinates for a spherically symmetric potential. We can transform the tidal tensor, which is the negative Hessian operator on a scalar function, into the same negative Hessian matrix in spherical coordinates. (This kind of transformation requires the use of tensor algebra, covariant differentiation, and a somewhat complicated and long calculation with differential geometric methods as shown in the appendix.) The tidal tensor for a spherical potential is

$$\tau^{\text{sph}} = \begin{pmatrix} -\frac{\partial^2 \Phi^{\text{sph}}(r)}{\partial r^2} & 0 & 0 \\ 0 & -\frac{1}{r} \frac{\partial \Phi^{\text{sph}}(r)}{\partial r} & 0 \\ 0 & 0 & -\frac{1}{r} \frac{\partial \Phi^{\text{sph}}(r)}{\partial r} \end{pmatrix} \quad (14)$$

In spherical coordinates the trace elements of τ^{sph} highlight the distinction between a tidal force induced by a radial change of the magnitude of the force field and that due to an angular one

$$\tau_{rr}^{\text{sph}} = -\frac{\partial^2 \Phi^{\text{sph}}(r)}{\partial r^2} = \frac{\partial \mathbf{F}(r)}{\partial r} \quad (15a)$$

$$\tau_{\theta\theta}^{\text{sph}} = \tau_{\phi\phi}^{\text{sph}} = F_{\theta}(r) = F_{\phi}(r) = -\frac{1}{r} \frac{\partial \Phi^{\text{sph}}(r)}{\partial r} = \frac{\mathbf{F}(r)}{r}. \quad (15b)$$

The first and second derivatives with respect to r of Eq. (11) are

$$\frac{\partial \Phi^{\text{sph}}(r)}{\partial r} = \frac{4\pi G}{r^2} \int_0^r \rho(r')(r')^2 dr' = \frac{GM(r)}{r^2} \quad (16)$$

$$\frac{\partial^2 \Phi^{\text{sph}}(r)}{\partial r^2} = -\frac{8\pi G}{r^3} \int_0^r \rho(r')(r')^2 dr' + 4\pi G\rho(r) = -\frac{2GM(r)}{r^3} + 4\pi G\rho(r). \quad (17)$$

Newton's second theorem for the force field of a spherical potential emerges automatically

$$F = -\frac{\partial \Phi^{\text{sph}}(r)}{\partial r} = -\frac{GM(r)}{r^2}. \quad (18)$$

The direct application of Newton's second theorem is sufficient to determine the force field, and no explicit evaluation of the potential derivatives is necessary. However, when dealing with tidal forces in the presence of local matter where $\frac{\partial M(r)}{\partial r} \neq 0$, we cannot avoid the explicit

evaluation of both derivatives and we obtain

$$\tau^{\text{sph}} = \begin{pmatrix} \frac{2GM(r)}{r^3} - 4\pi G\rho(r) & 0 & 0 \\ 0 & -\frac{GM(r)}{r^3} & 0 \\ 0 & 0 & -\frac{GM(r)}{r^3} \end{pmatrix}. \quad (19)$$

Unlike gravitational force fields it is important to keep in mind that tidal fields depend on the inside and the outside matter distributions. If we define $\bar{\rho}(r) = \frac{M(r)}{\frac{4}{3}\pi r^3}$ as the enclosed medium matter density, that is, the homogenous density that would result from a mass $M(r)$ enclosed in the sphere of radius r , we can also write

$$\tau^{\text{sph}} = 4\pi G \begin{pmatrix} \frac{2}{3}\bar{\rho}(r) - \rho(r) & 0 & 0 \\ 0 & -\frac{1}{3}\bar{\rho}(r) & 0 \\ 0 & 0 & -\frac{1}{3}\bar{\rho}(r) \end{pmatrix} \quad (20)$$

Thus we can also say that the strength of the radial tidal forces is proportional to the deviation of the local density from the enclosed medium matter density $\bar{\rho}(r)$.

As expected, the trace of τ^{sph} is the scalar $-4\pi G\rho(r)$, according to Poisson's equation. However, the tidal tensor furnishes more information than Poisson's equation alone, because it also gives the individual contributions along the three coordinate axes. The components of τ^{sph} depend on the local mass density. The angular variation in $d\theta$ or $d\phi$ leads to a corresponding directional change of the tidal force vector, which, in a spherical mass distribution, is a movement along iso-density lines.

III. THE PHENOMENON OF THE TIDAL FORCE REVERSAL: THE SPHERICAL MASS DISTRIBUTION

A. The tidal field for a general mass distribution model and for some special cases

The spherical heterogeneous mass distribution shows an interesting possibility that cannot occur in the traditional case of a point mass or for tidal fields outside mass distributions. For special conditions the tidal radial component τ_{rr}^{sph} in Eqs. (14), (19), or (20), can change its sign. Moving radially inside the mass distribution, the tensile radial tidal force (the force responsible for the usual tidal bulges) can become weaker, eventually become zero, and even change its sign, that is, become a compressive radial tidal force. For the spherical

mass distribution this possibility occurs when the local density nears, equals, or becomes greater than the enclosed medium matter density. In the next sections we will show how this behavior is a feature of tidal forces inside mass distributions for many geometries.

The gravitational force never changes sign, but its derivative can, and that can make a difference for tidal forces. It all depends on how the local matter density function behaves: in some physical conditions we may well have an increase of gravitational forces moving away from the central regions.

We now study this aspect more closely for the spherical heterogeneous mass distribution. In particular, let us examine the sign reversal of the tensor component $\tau_{rr}^{\text{sph}} = -\partial^2\Phi^{\text{sph}}(r)/\partial r^2 = \partial F(r)/\partial r$, and see what it represents. From Eq. (17) we see that a transition from a tensile to a compressive radial force occurs when

$$\tau_{rr}^{\text{sph}} = -\frac{\partial^2\Phi^{\text{sph}}(r)}{\partial r^2} = \frac{8\pi G}{r^3} \int_0^r \rho(r')(r')^2 dr' - 4\pi G\rho(r) = 0. \quad (21)$$

An immediate solution is the case of the singular mass density function, $\rho(r) \propto 1/r$, which leads to the exact equality of Eq. (21) for any radial distance r . Equation (21) implies that a spherical mass distribution with a mass density function proportional to the inverse of the distance from its center exactly cancels the radial tidal forces everywhere. For any radial movement the increase (decrease) of the mass enclosed in a sphere of radius r is such that it determines an increase (decrease) of the force field that exactly balances its decrease (increase) because of this radial change: the net gravitational force field is constant inside the whole mass distribution. The potential $\Phi(r) \propto \Phi_0 + cr$, with $\Phi_0 \equiv \Phi(0)$ and c an arbitrary constant, the force $\mathbf{F} = \partial\Phi(r)/\partial r \propto c = \text{constant}$, and there is no gradient of the force field, $\partial^2\Phi/\partial r^2 \equiv 0$, that is, no tidal forces anywhere. In principle, we could imagine multiple star systems, Oort clouds, or entire star clusters moving inside such a mass distribution without being affected by a tidal perturbation.

The mass density function $\rho(r) \propto 1/r$ is interesting as a theoretical limit but is unrealistic or at least not very common in astrophysical contexts.²¹ It is a theoretical limiting case between a situation that gives rise to tensile or compressive tidal forces.

For a mass distribution function $\rho(r) \propto 1/r^\gamma$, Eq. (21) becomes

$$\tau_{rr}^{\text{sph}} = -\frac{\partial^2\Phi^{\text{sph}}(r)}{\partial r^2} \propto \frac{4\pi G}{r^\gamma} \frac{\gamma - 1}{3 - \gamma} \leq 0, \quad (22)$$

which is satisfied if and only if $\gamma \leq 1$. Inside a spherical mass distribution with $\gamma \leq 1$, the radial tidal forces are always compressive. The usual tensile radial tides with their induced

tidal bulges to which we are accustomed no longer exist. One example is the homogeneous mass distribution for $\gamma = 0$, $\rho(r) = \rho_0 = \text{constant}$, for which we obtain

$$\tau_{rr}^{\text{sph}} = -\frac{\partial^2 \Phi^{\text{sph}}(r)}{\partial r^2} = -\frac{4\pi}{3}G\rho_0. \quad (23)$$

In this case it is not the force field that is constant (it grows linearly with r), but the radial tidal component, which is a compression force.

For $\gamma < 1$, $M(r) \propto r^\alpha$ with $\alpha > 2$, that is, the rate of increase of the inner mass enclosed in a sphere of growing radius r is faster than the inverse of the square root law. If gravitational forces increase with r , the derivative has the opposite sign from what we are accustomed to seeing for external or pointlike masses: it is this fact that leads to the tidal reversal which is nonexistent for any field external to a mass distribution. We will see that it is a common occurrence in the central regions of globular clusters, elliptic galaxies, and galaxy bulges. Moreover, the notion of Roche's tidal radius is meaningless. Moving closer to this tidal reversal zone, the tidal radius diverges toward infinity, and beyond this limit it does not exist. This analytic divergence of the tidal radius should be taken into account when we consider stellar mass distributions, which if neglected can lead to numerical errors.

B. The tidal field for the isothermal mass distribution model

So far we have analyzed some special cases that are unlikely in the real world (though not completely unphysical). We will now consider cases with $\gamma > 0$,²² where the central density does not go to infinity.

An interesting case of a real spherical heterogeneous mass distribution is that of globular clusters.²³ The surface brightness of these objects is well approximated by King's light curve law.²⁵ Therefore, if we assume that the visible bright matter represents the real matter content (globular star clusters have a very low dark matter content if any), King's surface brightness can be recovered by a three-dimensional isothermal mass distribution¹³

$$\rho_{\text{iso}}(r) = \frac{\rho_0}{1 + (r/r_c)^2} = \frac{\rho_0}{1 + \eta^2}, \quad (24)$$

where ρ_0 is the central density, r_c a core radius (defined as the radius where the surface brightness is 50% of the central luminosity and is typically about the order of a parsec in globular clusters and from 10^2 to 10^3 pc for galactic bulges), and $\eta = r/r_c$ is a scale length.²⁴

The distribution ρ_{iso} is of order $1/r^\gamma$ with $\gamma \sim 0$ in the central regions and $\gamma \sim 2$ for the external ones. Thus, from what we have seen in Sec. III A, we can expect a tidal reversal phenomenon close to the central region.

We substitute Eq. (24) into Eqs. (11) and (12) and obtain the potential and the enclosed mass at the radius r for the isothermal mass distribution. If we scale all formulas by the scale length $\eta = r/r_c$, we obtain

$$\Phi_{\text{iso}}(\eta) = -4\pi G r_c^2 \rho_0 \left[\frac{\eta - \arctan \eta}{\eta} - \frac{1}{2} \log \left(\frac{1 + \eta^2}{1 + \eta_{\text{max}}^2} \right) \right]. \quad (25)$$

We use Eqs. (12), (14), and (18) for the gravitational force field and the tidal tensor elements and obtain

$$F_{\text{iso}}(\eta) = -4\pi G \rho_0 r_c \frac{(\eta - \arctan \eta)}{\eta^2} \quad (26)$$

$$\tau_{rr}^{\text{iso}}(\eta) = 4\pi G \rho_0 \left[\frac{2(\eta - \arctan \eta)}{\eta^3} - \frac{1}{1 + \eta^2} \right] \quad (27)$$

$$\tau_{\theta\theta}^{\text{iso}}(\eta) = \tau_{\phi\phi}^{\text{iso}}(\eta) = -4\pi G \rho_0 \frac{(\eta - \arctan \eta)}{\eta^3}. \quad (28)$$

We see that $\lim_{\eta \rightarrow 0} \Phi_{\text{iso}}(\eta) = -2\pi G r_c^2 \rho_0 \log(1 + \eta_{\text{max}}^2)$ is finite if we assume there is a cutoff radius $R_{\text{max}} = r_c \eta_{\text{max}}$ for the dimension of the globular cluster. Similarly, $M_{\text{iso}}(\eta_{\text{max}}) = 4\pi \rho_0 r_c^3 (\eta_{\text{max}} - \arctan \eta_{\text{max}})$ is finite and represents the cluster's total mass. There is no singularity for Eq. (26) because $\lim_{\eta \rightarrow 0} (\eta - \arctan \eta)/\eta^2 = 0$, which is consistent with the fact that gravitational forces must disappear at the center of a spherical mass distribution. Note how for large η it converges toward an inverse force law instead of an inverse square one.

What do the gravitational and the tidal force fields look like in an isothermal mass distribution? In Fig. 1 we show plots of Eqs. (27) (continuous line) and (28) (dashed line), both normalized to $4\pi G \rho_0$, versus the scale length η . The gravitational force field increases with distance from the center until $\sim 1.5\eta$. The radial tidal field is correspondingly negative, which means that there are radial forces of compression in this internal region. From Fig. 1 we see that there is a relatively intense negative peak for the radial tidal field in the central part compared with the positive maxima at about $\sim 2.7\eta$. From this behavior it is clear how in the central regions of an isothermal sphere the radial tidal compression is much greater than the tensile ones. From Eqs. (27) and (28) we find that all the tidal tensor elements at the center of the cluster (that is, for $\eta \rightarrow 0$) have a common value of $-4\pi G \rho_0/3$.

Compared with the positive maxima at about $\eta \sim 2.7$, which does not exceed $\sim 0.03 \times 4\pi G\rho_0$, it is clear how the compressive radial tidal effects in a globular cluster's center might be more than ten times greater than any radial tensile tidal force anywhere else. Moreover, it is clear that there must be a small region inside such a mass distribution at about ~ 1.5 scale lengths where the radial tidal effects are zero, beyond which the negative tides take over abruptly.

C. The tidal field for the Plummer sphere mass distribution model

We briefly repeat the same calculations for the Plummer sphere model¹³ which is sometimes used as a first approximation for the description of the internal structure of galactic bulges. Plummer's model is interesting because it describes the simplest spherical potential that does not diverge in the central regions as

$$\Phi_{\text{Pl}}(r) = -\frac{GM_T}{\sqrt{r^2 + r_c^2}}, \quad (29)$$

where M_T is the total mass. To recover the mass density function, $\rho_{\text{Pl}}(r)$, we need to apply Poisson's equation directly

$$\nabla^2 \Phi_{\text{Pl}}(r) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi_{\text{Pl}}}{dr} \right) = 4\pi G \rho_{\text{Pl}}(r). \quad (30)$$

Hence

$$\rho_{\text{Pl}}(r) = \frac{\rho_0}{(1 + \eta^2)^{5/2}}, \quad (31)$$

with $\rho_0 = 3M_T/4\pi r_c^3$. By using Eqs. (12), (14), and (18) we see again that the gravitational force field and the tidal tensor elements are

$$F_{\text{Pl}}(\eta) = -\frac{4}{3}\pi G \rho_0 r_c \frac{\eta}{(1 + \eta^2)^{3/2}} \quad (32)$$

$$\tau_{rr}^{\text{Pl}}(\eta) = \frac{4}{3}\pi G \rho_0 \frac{2\eta^2 - 1}{(1 + \eta^2)^{5/2}} \quad (33)$$

$$\tau_{\theta\theta}^{\text{Pl}}(\eta) = \tau_{\phi\phi}^{\text{Pl}}(\eta) = -\frac{4}{3}\pi G \rho_0 \frac{1}{(1 + \eta^2)^{3/2}}. \quad (34)$$

Plots of the tidal force fields are shown in Fig. 2. Its similarity with the isothermal mass distribution is evident, but the gravitational force field is peaked more toward its center and consequently there is a much more marked abrupt tidal reversal. Therefore it is expected that in galactic centers the effect is less smooth than in globular clusters.

IV. CONCLUSION AND POSSIBLE DIRECTIONS OF FUTURE RESEARCH

We have seen that expressing tidal forces in terms of a tensor formalism naturally leads to understanding some features of a tidal field inside mass distributions. As an example of its possible uses, we applied it to some astrophysical relevant cases. We saw how an inverse radial tidal compressive effect emerges in the central regions of an isothermal model and a Plummer model, which are first approximations for globular star clusters and galactic bulges mass distributions.²⁸ In these models the intensity of the central compressive tidal effects depend only on the central density ρ_0 , and the dimension of this central region is of the order of the core radius. The most intense compressive tidal effects are expected for globular clusters or galactic bulges with a relatively big core radius and a high central density.

These compressive tides may lead to dynamical effects (as might have occurred in the evolutionary history of stellar clusters) in the form of gravitational tidal shocks,⁶ in tidal heating,⁷ in star forming clouds,⁸ in binary or multiple star systems, in the Oort clouds of comets,²⁶ and is also expected in open star clusters when their orbital path went through this internal region. For example, Oort clouds are dynamic systems that can be very sensitive to the external galactic tides.⁵ Galactic radial compressive tides can reduce the Oort cloud volume considerably if it is inside the radial galactic tidal reversal zone. But this compression would not lead to any gravitational collapse because the Oort cloud comets can be considered as a non-self-interacting system of particles. Very different and perhaps dramatic effects would cause a volume reduction of an interstellar cloud leading to a gravitational collapse and to corresponding star forming process. There is now sufficient evidence that suggests how some still to be determined, yet very intense star forming processes occur in the center of galaxies up to the present. The tidal reversal phenomenon that occurs in the core radius regions might be part of a possible explanation, despite the existence of massive black holes whose disruptive tides would prevail only a few light years from its center. (Therefore, we would expect a ring shaped star forming region.) Also, the transient and abrupt tidal compressions we have seen at about the core radius might be considered as a source for a Jeans instability.²⁷

Isothermal models are also believed to approximate the dark matter halos of galaxies. A galactic dark halo has a much lower central density (the order of $\sim 0.1M_\odot/pc^3$) than

those found in globular clusters or galaxy centers ($\sim 1000M_\odot/pc^3$), and therefore any tidal contribution of a halo can be neglected. It is tempting to conjecture that during the first cosmological phases, where the non-baryonic dark matter density in the universe was much higher, if not dominant over bright matter, a compressive instead of a disruptive tidal effect could have played a role. This effect might have ignited a star formation process in some regions of the universe much sooner than previously expected, as has been observed at high redshifts. These lines of research need further study.

Acknowledgments

I would like to thank Chris Hillman for his valuable suggestions on how to calculate the tidal tensor in other coordinate systems.

APPENDIX: THE HESSIAN OPERATOR FOR A SPHERICAL COORDINATE SYSTEM

The nature of the Christoffel symbols necessary to calculate covariant derivatives can be found in many textbooks. However these textbooks rarely cover the case of the generalized fully covariant connection coefficients that are necessary to obtain the Hessian in spherical coordinates. This need is an opportunity to recapitulate covariant differentiation in a broader context than usually given in introductory courses on differential geometry.

If $\{x_1, x_2, x_3\}$ represent the Cartesian coordinate system and $\{u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3)\}$ represent a new one, then we might expect that by applying the usual transformation law for a second rank tensor, the Hessian matrix of a scalar function Φ , $H_{\alpha\beta} = \partial^2\Phi/\partial x_\alpha\partial x_\beta$, in these new set of coordinates should become²⁹

$$H'_{\alpha\beta} = \frac{\partial^2\Phi}{\partial u_\alpha\partial u_\beta} = H^{\gamma\delta} \frac{\partial x_\gamma}{\partial u_\alpha} \frac{\partial x_\delta}{\partial u_\beta}. \quad (\text{A.1})$$

Unfortunately, in general $\partial^2\Phi/\partial u_\alpha\partial u_\beta$ is not a tensor because this transformation does not take into account the fact that the Hessian is the second derivative of the gradient, which is not a vector but a covector. As it is known from tensor analysis, to differentiate a covector requires covariant differentiation. Thus, to recover the tidal tensor (the negative Hessian of the potential) for other coordinate systems, we need to first calculate the covariant derivative

of the gradient (written also in the new coordinates).

The covariant derivative (usually labeled by the symbol “;”) of a covector T is

$$T_{\beta;\gamma} = T_{\beta,\gamma} - \Gamma_{\beta\gamma}^{\alpha} T_{\alpha}, \quad (\text{A.2})$$

where $T_{\beta,\gamma}$ is the normal “comma derivative” on the component γ of T , and $\Gamma_{\beta\gamma}^{\alpha}$ are the connection coefficients of the second kind. For Cartesian coordinates all the $\Gamma_{\beta\gamma}^{\alpha}$ vanish and the covariant derivative becomes the usual one. In this case the covector is the gradient of Φ in the new coordinates $T_{\gamma} = \partial\Phi/\partial u_{\gamma}$.

To calculate the connection $\Gamma_{\alpha\beta}^{\gamma}$ we need to distinguish between coordinate bases (also called holonomic bases) and non-coordinate bases (anholonomic bases). It is beyond the scope of this paper to give a detailed and rigorous description of the differential geometric meaning of these complications, and the interested reader is read the relevant literature (for example, Ref. 14, Secs. 8.4 and 8.5). However, we can clarify the difference briefly as follows.

Take the mathematician’s viewpoint that tangent vectors and directional derivatives are the same thing, $\mathbf{u} = \partial_{\mathbf{u}}$. Let \mathbf{u} and \mathbf{v} be two vector fields and define their commutator as

$$[\mathbf{u}, \mathbf{v}] \equiv [\partial_{\mathbf{u}}, \partial_{\mathbf{v}}] = \partial_{\mathbf{u}}\partial_{\mathbf{v}} - \partial_{\mathbf{v}}\partial_{\mathbf{u}}. \quad (\text{A.3})$$

For any basis $\{\mathbf{e}_{\alpha}\}$ we define the commutation coefficients $c_{\alpha\beta}^{\gamma}$ by

$$[\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}] = c_{\alpha\beta}^{\gamma} \mathbf{e}_{\gamma}. \quad (\text{A.4})$$

By definition a basis is a “coordinate basis” or holonomic if all the $c_{\alpha\beta}^{\gamma} = 0$ and a “noncoordinate basis” (anholonomic) if at least one $c_{\alpha\beta}^{\gamma} \neq 0$.

For noncoordinate bases we calculate the connection coefficients as $\Gamma_{\beta\gamma}^{\alpha} = g^{\alpha\mu} \Gamma_{\mu\beta\gamma}$, with $\Gamma_{\mu\beta\gamma}$ the fully covariant connection coefficient of the second kind given by

$$\Gamma_{\mu\beta\gamma} = \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu}), \quad (\text{A.5})$$

where $c_{\beta\gamma\alpha} = g_{\alpha\mu} c_{\beta\gamma}^{\mu}$.

For Cartesian coordinates $\mathbf{e}_x = \frac{\partial}{\partial x}$, $\mathbf{e}_y = \frac{\partial}{\partial y}$, $\mathbf{e}_z = \frac{\partial}{\partial z}$, all commutation coefficients vanish, that is, it is a coordinate basis, and Eq. (A.5) simplifies to the usual Christoffel symbols. For spherical coordinates

$$\mathbf{e}_{\hat{r}} = \frac{\partial}{\partial r}, \quad \mathbf{e}_{\hat{\theta}} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \quad (\text{A.6})$$

Ref. 14 obtain (p. 207)

$$c_{\hat{r}\hat{\theta}}^{\hat{\theta}} = -c_{\hat{\theta}\hat{r}}^{\hat{\theta}} = -\frac{1}{r}, \quad (\text{A.7})$$

$$c_{\hat{r}\hat{\phi}}^{\hat{\phi}} = -c_{\hat{\phi}\hat{r}}^{\hat{\phi}} = -\frac{1}{r}, \quad (\text{A.8})$$

$$c_{\hat{\theta}\hat{\phi}}^{\hat{\phi}} = -c_{\hat{\phi}\hat{\theta}}^{\hat{\phi}} = -\frac{\cot \theta}{r}. \quad (\text{A.9})$$

The connection coefficients for the spherical noncoordinate basis given by Eq. (A.5) are

$$\Gamma_{\hat{\theta}\hat{\theta}}^{\hat{r}} = -\Gamma_{\hat{r}\hat{\theta}}^{\hat{\theta}} = -\frac{1}{r}, \quad (\text{A.10a})$$

$$\Gamma_{\hat{\phi}\hat{\phi}}^{\hat{r}} = -\Gamma_{\hat{r}\hat{\phi}}^{\hat{\phi}} = -\frac{1}{r}, \quad (\text{A.10b})$$

$$\Gamma_{\hat{\phi}\hat{\phi}}^{\hat{\theta}} = -\Gamma_{\hat{\theta}\hat{\phi}}^{\hat{\phi}} = -\frac{\cot \theta}{r}, \quad (\text{A.10c})$$

and all other coefficients vanish.

To calculate the second derivatives of Φ (the Hessian), we must calculate the covariant derivatives (A.2). If we write $H_{\beta\gamma}(\Phi) = \frac{\partial^2 \Phi}{\partial u_\beta \partial u_\gamma} = T_{\beta;\gamma}$, with $T = \nabla \Phi$ the covariant vector to derive, then

$$H_{\beta\gamma}(\Phi) = (\nabla \Phi)_{\beta;\gamma} - \Gamma_{\beta\gamma}^\alpha (\nabla \Phi)_\alpha, \quad (\text{A.11})$$

with $\Gamma_{\beta\gamma}^\alpha$ the coefficients (A.10) in the basis (A.6). We obtain

$$H_{\hat{r}\hat{r}}(\Phi) = (\nabla \Phi)_{\hat{r};\hat{r}} - \Gamma_{\hat{r}\hat{r}}^\alpha (\nabla \Phi)_\alpha = \frac{\partial^2 \Phi}{\partial r^2} \quad (\text{A.12a})$$

$$H_{\hat{r}\hat{\theta}}(\Phi) = H_{\hat{\theta}\hat{r}}(\Phi) = (\nabla \Phi)_{\hat{r};\hat{\theta}} - \Gamma_{\hat{r}\hat{\theta}}^\alpha (\nabla \Phi)_\alpha = \frac{1}{r} \frac{\partial^2 \Phi}{\partial \theta \partial r} - \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} \quad (\text{A.12b})$$

$$H_{\hat{r}\hat{\phi}}(\Phi) = H_{\hat{\phi}\hat{r}}(\Phi) = (\nabla \Phi)_{\hat{r};\hat{\phi}} - \Gamma_{\hat{r}\hat{\phi}}^\alpha (\nabla \Phi)_\alpha = \frac{1}{r \sin \theta} \frac{\partial^2 \Phi}{\partial \phi \partial r} - \frac{1}{r^2 \sin \theta} \frac{\partial \Phi}{\partial \phi} \quad (\text{A.12c})$$

$$H_{\hat{\theta}\hat{\theta}}(\Phi) = (\nabla \Phi)_{\hat{\theta};\hat{\theta}} - \Gamma_{\hat{\theta}\hat{\theta}}^\alpha (\nabla \Phi)_\alpha = \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} \quad (\text{A.12d})$$

$$H_{\hat{\theta}\hat{\phi}}(\Phi) = H_{\hat{\phi}\hat{\theta}}(\Phi) = (\nabla \Phi)_{\hat{\theta};\hat{\phi}} - \Gamma_{\hat{\theta}\hat{\phi}}^\alpha (\nabla \Phi)_\alpha = \frac{1}{r^2 \sin \theta} \frac{\partial^2 \Phi}{\partial \phi \partial \theta} - \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial \Phi}{\partial \phi} \quad (\text{A.12e})$$

$$H_{\hat{\phi}\hat{\phi}}(\Phi) = (\nabla \Phi)_{\hat{\phi};\hat{\phi}} - \Gamma_{\hat{\phi}\hat{\phi}}^\alpha (\nabla \Phi)_\alpha = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial \Phi}{\partial \theta} \quad (\text{A.12f})$$

The Hessian matrix in spherical coordinates is written as

$$H(\Phi(r, \theta, \phi)) = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial r^2} & \frac{1}{r} \frac{\partial^2 \Phi}{\partial \theta \partial r} - \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial^2 \Phi}{\partial \phi \partial r} - \frac{1}{r^2 \sin \theta} \frac{\partial \Phi}{\partial \phi} \\ \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} & \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} & \frac{1}{r^2 \sin \theta} \frac{\partial^2 \Phi}{\partial \phi \partial \theta} - \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial \Phi}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial^2 \Phi}{\partial r \partial \phi} - \frac{1}{r^2 \sin \theta} \frac{\partial \Phi}{\partial \phi} & \frac{1}{r^2 \sin \theta} \frac{\partial^2 \Phi}{\partial \theta \partial \phi} - \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial \Phi}{\partial \phi} & \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial \Phi}{\partial \theta} \end{pmatrix} \quad (\text{A.13})$$

Note that the trace of H equals the Laplacian of Φ in spherical coordinates.

In general, Eq. (A.13) is not a particularly useful or elegant expression for the Hessian. However, if there is spherical symmetry, this matrix simplifies and the tidal tensor, which is the negative Hessian of the gravitational potential Φ , $\tau = -H$, applied to a spherical potential Φ^{sph} becomes the diagonal matrix (14), as expected (all angular derivatives $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi}$ in Eq. (A.13) vanish).

If we are interested in describing tidal fields in elliptic or disk-like galaxies, we can repeat the above calculation to obtain the Hessian in elliptic and cylindrical coordinates respectively.

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¹ For a rigorous and detailed description of tides occurring in the solar system see for example, S. F. Dermott and C. D. Murray, *Solar System Dynamics* (Cambridge University Press, Cambridge, 1999).

² There is a large literature on the subject. Two introductory references are J. E. Barnes, “Encounters of disk/halo galaxies,” *Astron. J.* **331**, 699–717 (1988), and *Galaxies: Interactions and Induced Star Formation*, edited by R. C. Kennicutt Jr., F. Schweizer, and J. E Barnes, Saas-Fee Advanced Course 26, Swiss Society for Astrophysics and Astronomy (Springer, Berlin, 1998).

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- ⁹ S. D. M. White, “Angular momentum growth in protogalaxies,” *Astrophys. J.* **286**, 38–41 (1984).
- ¹⁰ C. Porciani, A. Dekel, and Y. Hoffman, “Testing tidal-torque theory: I. Spin amplitude and direction - Testing tidal-torque Theory: II. Alignment of inertia and shear and the characteristics of proto-haloes,” *MNRAS* **332**, 325–351 (2002).
- ¹¹ In some of the literature the tidal tensor is identified only with the pure shear deformation, that is, the traceless part of Eq. (3), $\tau_{ij} - \frac{\delta_{ij}}{3}\tau_{ii}$, and Eq. (3) is called the “deformation tensor.” In general, tidal deformations can be a combination of both shear and transverse stretch. We will maintain the convention where both stretches are considered because, as we will see, it is more consistent with the tidal tensor of general relativity.
- ¹² When considering a generic, Cartesian, or radial coordinate system we will write $\Phi(\mathbf{R})$, $\Phi(x, y, z)$, or $\Phi(r)$ accordingly.
- ¹³ J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1988).
- ¹⁴ C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
- ¹⁵ H. Bondi, “The foundations of the theory of gravitation,” *Acta Physica Polonica B* **30** (10), 2859–2862 (1999).
- ¹⁶ We will use forces and potentials per unit mass.
- ¹⁷ From a purely algebraic point of view, we should in principle calculate the eigenvalues of τ because every $n \times n$ real symmetric matrix is orthogonally similar to a diagonal matrix whose diagonal elements are its eigenvalues. Fortunately the diagonalization, at least in this case, is much simpler.
- ¹⁸ Some authors prefer to use the opposite sign. It is a matter of convention. We follow this convention because it emerges directly from the calculations without ambiguity.
- ¹⁹ R. Caimmi and L. Secco, “Around the Newton’s theorem for ellipsoidal two-component figures,” *Astr. Gesellschaft Jenam Abstract Series*, **18**, 211 (2001).
- ²⁰ Newton tried, without success, to show that this theorem also holds for homeoids (ellipsoidal shells of matter), but the rigorous proof came about two centuries later.¹³ Interesting applications and generalizations have been done recently also in Ref. 19.
- ²¹ On the other side $\lim_{r \rightarrow 0} \rho(r) = \infty$ does not make this case unphysical if we think of the existence of a singularity as a black hole at the center of the star cluster, and especially notice

how the mass does not diverge, because $\lim_{r \rightarrow 0} M(r) = \lim_{r \rightarrow 0} (2\pi cr^2) = 0$.

- ²² The case $\gamma < 0$ on an interval in r implies an increasing density with r which, even if not physically impossible (for example ring-shaped matter distributions), is unlikely, or at least much less frequently in nature, and we will not address it here.
- ²³ A similar case has been studied by Valluri,⁷ but without transforming the tidal tensor to spherical coordinates. We will compare this profile with Plummer’s in Sec. III C.
- ²⁴ This density distribution corresponds to a stable arrangement of a self-gravitating system that satisfies the equations of hydrostatic equilibrium of an ideal gas at a constant temperature.
- ²⁵ I. R. King, “Surface photometry of elliptical galaxies,” *Astrophys. J.* **222** Part 1, 1–13 (1978).
- ²⁶ M. Masi, “Dynamical effects of the radial galactic tide on an Oort cloud of comets for stars with different masses and varying distances from the galactic center,” *astro-ph/0403599*. This preprint does not cover the case of the inverse tidal effects.
- ²⁷ The Jeans instability occurs when the thermal pressure of a cloud of stellar gas, which causes the cloud to expand, is no longer large enough to balance the gravitational pressure that causes the cloud to collapse. In the scenarios we describe here we must consider not only the self-gravitational pressure, but the tidal pressure must also be taken into account.
- ²⁸ The models we have considered here are only approximations; more realistic mass distribution might give more precise answers. However, the use of these approximations does not undermine the fact that the tidal reversal is not an exception for a specific model, but is a universal feature of the gravitational field.
- ²⁹ Einstein’s summation convention is used.

FIGURE CAPTIONS

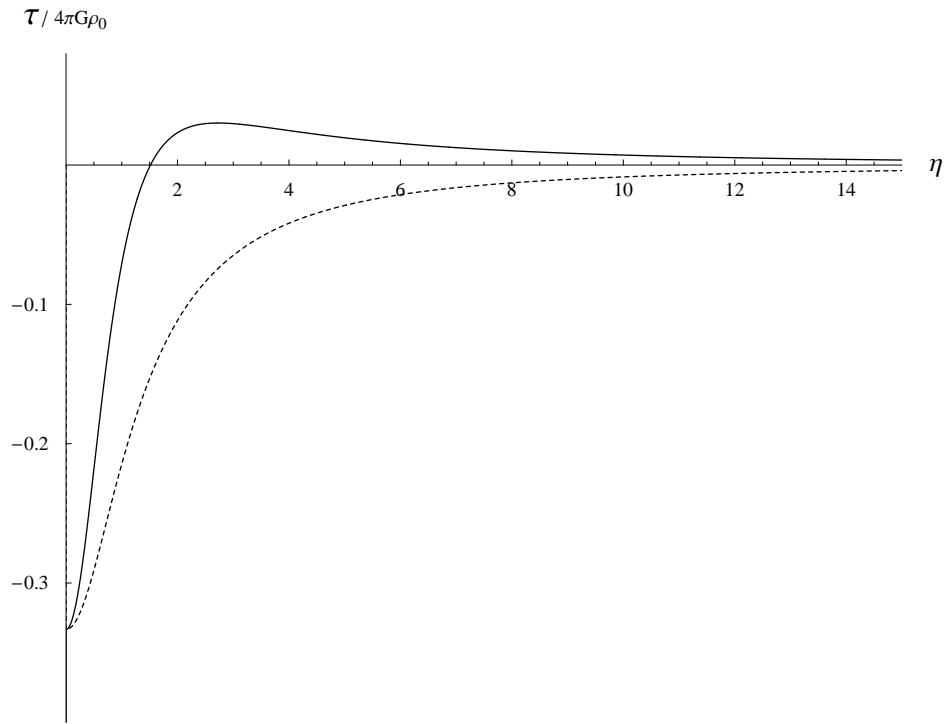


FIG. 1: The tidal force field of the isothermal spherical mass distribution $\tau_{rr}^{\text{iso}}(\eta)$ (continuous line) and $\tau_{\theta\theta}^{\text{iso}}(\eta) = \tau_{\phi\phi}^{\text{iso}}(\eta)$ (dashed line).

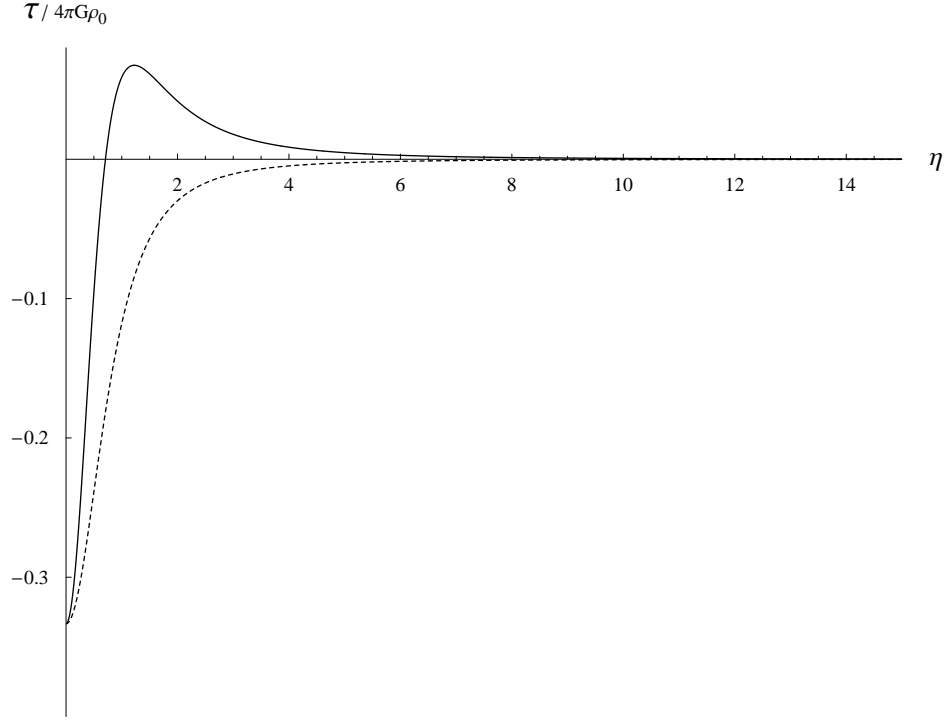


FIG. 2: The tidal force field of the Plummer spherical mass distribution $\tau_{rr}^{\text{Pl}}(\eta)$ (continuous line) and $\tau_{\theta\theta}^{\text{Pl}}(\eta) = \tau_{\phi\phi}^{\text{Pl}}(\eta)$ (dashed line).